

TCFT ELLIPTIC OBJECTS

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ABSTRACT. This paper begins to develop a topological conformal field theory analogue of the Segal-Stolz-Teichner project for geometric construction of elliptic cohomology. In our context, we replace the theory Von-Neuman algebra bimodules and Connes fusion by differential graded category bimodules, and their natural categorical tensor product. At the same time, we extend the notion of bicategory in a geometrically natural fashion, and using this we show that some functors on the homotopy category of topological spaces coming from topological conformal field theory are representable.

1. INTRODUCTION

We begin to develop here a topological conformal field theory analogue of the fascinating proposal in Stolz-Teichner [7], following Segal [6], for geometric construction of tmf -cohomology (a kind of universal elliptic cohomology) in terms of enriched elliptic Segal objects. The present note is only meant to complement the work of Stolz-Teichner; morally we only look at part of the data that Stolz-Teichner consider and from a very different point of view at that. In particular although it is possible that our TCFT variant gives a new multiplicative cohomology theory (at least after incorporating more data, see Remark 3.2), it doesn't look like tmf -cohomology. The hope however is that it should be closely related. In particular our construction produces a complicated graded ring, that we would like to relate to $\mathrm{tmf}(pt)$. Moreover some of the ideas here should be helpful in the original proposal of Stolz-Teichner, particularly the notion of a relative bicategory, which at least from our perspective here seems indispensable. The main result here is the construction of a series of representable functors, constructed from topological conformal field theory, which is a significant step toward showing existence of a generalized cohomology theory based on conformal field theory.

The formal properties of the elliptic objects of Segal-Stolz-Teichner are heavily motivated by properties of a specific elliptic object that can be assigned to any string vector bundle over a space X , this is the so called “elliptic Euler class”. This object arises naturally via spin geometry. Likewise we are also motivated by a “god given” TCFT object, which can be assigned to “any” Hamiltonian fiber bundle over a space X with a connection, for example to a bundle of Calabi-Yau manifolds as is familiar to physicists, or to projectivization of a Hermitian vector bundle. This object arises via symplectic geometry, and we call it the Donaldson-Floer-Fukaya TCFT object, for the main contributors to the background needed for existence of this object. In particular, taking $X = pt$, this gives a 2-categorical enrichment of Gromov-Witten theory of a symplectic manifold. Details of construction of this geometric object, which are analytical are to appear in a sequel.

Let us now take a step back to describe some of the background for the work of Teichner and Stolz, although the introduction of [7] does a much better job

of it. Segal’s original vision of elliptic cohomology, (forgetting some structure) is roughly that it is derived from rings of equivalence classes of Top enriched tensor functors from the string category of X , whose objects are collections of loops in X and morphisms are 2d-bordisms with domain a conformal Riemann surface: $\Gamma : (\Sigma_g, j) \rightarrow X$, to the tensor category of topological vector spaces. Let us call as in [7] such a functor a Segal object. In this way Segal objects are highly analogous to geometric representatives of K -theory of X as continuous tensor functors from the “path category” of X to the unitary group considered as a topological category.

However as pointed out by Teichner-Stolz an interesting new difficulty arises for Segal objects in that Mayer-Vietoris property seems to fail: a pair of objects on U , V coinciding on intersection may not come from an object on $U \cup V$. For example it is not even clear how to construct the “Hilbert” space associated to a loop not completely contained in either U or V . The proposal of Teichner-Stolz to deal with this problem is essentially to have the entire closed string sector of conformal field theory on X be emergent from open string data. In this way it is reminiscent of the foundational work of Costello in [2] in the TCFT setting. Arguably the main technical ingredient in this proposal is the use of Von-Neumann algebra bi-modules and Connes fusion operation. The starting point for the present paper is the realization (probably not new) that Von-Neumann algebra bi-modules and Connes fusion also have an interesting formal analogue in the theory of topological conformal field theory: bi-modules over differential graded or A_∞ -categories and a fusion or composition operation, which from a correct categorical view point is just “tensor product”, but which in our particular case it is intimately related to Hochschild chain complex. This leads us to define a notion of TCFT elliptic objects as functors to a certain bicategory of small dg-categories, from a bicategory associated to X coming from open/closed Riemann surfaces. However for this to work well we actually have to naturally extend the notion of a bicategory.

When $X = pt$ our construction seems to have some connection to Costello’s [2], which was of course a great source of inspiration. At least indirectly we are also influenced by Lurie’s [5]. Some of the ideas here also seem related to Turaev’s Homotopy field theory, [8].

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2. PRELIMINARIES

Notation 2.1. *Note that we will always use diagrammatic order for composition of functors and morphisms i.e. the composition*

$$A \xrightarrow{f} B \xrightarrow{g} C,$$

is written as fg and this explains the order of the tensor products below. This will mean that the geometric “right” action is really right action in algebraic sense, when we come to bimodules below. Elsewhere, we use the standard Leibnitz convention.

Our categories and functors in all contexts are always assumed to be unital, which we sometimes emphasize explicitly.

2.1. Bicategories. A bicategory is a relaxation of the notion of a category enriched over the monoidal category of small categories. It is a collection of objects \mathcal{C}_0 , a collection of 1-morphisms $\mathcal{C}_1(a, b)$, for any pair of objects $a, b \in \mathcal{C}_0$, and a collection of 2-morphisms $\mathcal{C}_2(m_1, m_2)$ for $m_1, m_2 \in \mathcal{C}_1(a, b)$. There are composition maps

$$\mathcal{C}_1(a, b) \times \mathcal{C}_1(b, c) \rightarrow \mathcal{C}_1(a, c),$$

which are associative and unital up to natural 2-morphisms, called associator and unitor. The composition of 2-morphisms is required to be strictly associative and unital. Consequently, we get a category $\mathcal{C}(a, b)$ with objects $\mathcal{C}_1(a, b)$ and morphisms given by 2-morphisms. And there is a required “horizontal” composition operation of 2-morphisms, which is a functor

$$\mathcal{C}(a, b) \times \mathcal{C}(b, c) \rightarrow \mathcal{C}(a, c),$$

associative and unital up to natural transformation. There are usually in addition some imposed coherence identities on associators, unitors, similar to what is done for monoidal categories. We do not need this here (at least not yet), as it principally only plays a role in Mac Lane style coherence theorems.

Keeping this in mind, we will need to work with a slightly more general notion of a bicategory, which we now define. The generalization comes from need to allow source and target of 1-morphisms to change when related by a 2-morphism. The geometric motivation of this will eventually be completely apparent, so we just ask for patience on the part of the reader.

Definition 2.2. A **relative bicategory** (\mathcal{C}, D, L, R) consists of: a collection of objects \mathcal{C}_0 with an associated category D whose morphisms are called **boundary transformations**, a collection of 1-morphisms $\mathcal{C}_1(a, b)$, for any pair of objects $a, b \in \mathcal{C}_0$. A collection of 2-morphisms $\mathcal{C}_2(m_1, m_2)$ for $m_1, m_2 \in \mathcal{C}_1(a, b)$. There is a composition operation on 1-morphisms, which is associative and unital up to natural 2-morphisms. Likewise there is a **vertical composition** operation of 2-morphisms

$$\mathcal{C}_2(m_1, m_2) \times \mathcal{C}_2(m_2, m_3) \rightarrow \mathcal{C}_2(m_1, m_3),$$

where m_1, m_2, m_3 are not necessarily in the 1-morphism space of the same pair of objects $a, b \in \mathcal{C}_0$, and this composition operation is required to be unital and associative. In particular, there is an associated category \mathcal{C}_1 whose objects are 1-morphisms, (not the boundary transformations).

Finally, there are functors

$$L : \mathcal{C}_1 \rightarrow D,$$

$$R : \mathcal{C}_1 \rightarrow D$$

And a **horizontal composition** of 2-morphisms which is a functor

$$\mathcal{C}_1 \times_{L, R} \mathcal{C}_1 \rightarrow \mathcal{C}_1,$$

with composition law associative and unital up to natural transformations, where $\mathcal{C}_1 \times_{L, R} \mathcal{C}_1$ denotes the fibre product with respect to L, R .

Notation 2.3. Later on when there is no risk of confusion, we omit specifying the category of boundary transformations D for a relative bicategory, referring to it as we would to a usual bicategory.

Functors and natural transformations. When we say functor between bicategories we always mean 2-functor.

Definition 2.4. For a pair of relative bicategories

$$\mathcal{C} = (\mathcal{C}, D, L, R), \mathcal{C}' = (\mathcal{C}', D', L', R')$$

a functor $F : \mathcal{C} \rightarrow \mathcal{C}'$ is a straightforward generalization of a functor of bicategories. First it is a functor $F_{bd} : D \rightarrow D'$. It then assigns an object $F(a) \in \mathcal{C}'_0$ for every $a \in \mathcal{C}_0$ a morphism $F(m) \in \mathcal{C}'_1(F(a), F(b))$ for every $m \in \mathcal{C}_1(a, b)$, s.t. this assignment determines a functor:

$$F_1 : \mathcal{C}_1 \rightarrow \mathcal{C}'_1,$$

inducing a commutative diagram:

$$\begin{array}{ccc} \mathcal{C}_1 & \xrightarrow{F_1} & \mathcal{C}'_1 \\ \downarrow L, R & & \downarrow L', R' \\ D & \xrightarrow{F_{bd}} & D' \end{array}.$$

In particular F_1 induces a functor

$$\mathcal{C}_1 \times_{L, R} \mathcal{C}_1 \rightarrow \mathcal{C}'_1 \times_{L', R'} \mathcal{C}'_1$$

and we require that the following diagram commutes up to natural transformation:

$$(2.1) \quad \begin{array}{ccc} \mathcal{C}_1 \times_{L, R} \mathcal{C}_1 & \longrightarrow & \mathcal{C}_1 \\ \downarrow & & \downarrow \\ \mathcal{C}'_1 \times_{L', R'} \mathcal{C}'_1 & \longrightarrow & \mathcal{C}'_1 \end{array}.$$

Natural transformations of functors of relative bicategories will of course be crucial. It is perhaps instructive to first remind the reader how this works for usual bicategories. Let F_1, F_2 be a pair of functors between bicategories $\mathcal{C}_1, \mathcal{C}_2$. A natural transformation assigns 1-morphism $N(o) : F_1(o) \rightarrow F_2(o)$ for every $o \in \mathcal{C}_0$, and a 2-morphism $com(m)$ for every $m : o_1 \rightarrow o_2$ which is an isomorphism

$$com(m) : N(o_1) \circ_1 F_2(m) \rightarrow F_1(m) \circ_1 N(o_2).$$

The 2-morphism $com(m)$ is required to be natural, which we may elaborate as follows. Given $m, m' : o_1 \rightarrow o_2$ and $c : m \rightarrow m'$ in \mathcal{C}_2 , we have the following 2-morphism (with \circ_h denoting horizontal composition of 2-morphisms)

$$N(o_1) \circ F_2(m) \xrightarrow{id \circ_h F_2(c)} N(o_1) \circ F_2(m') \xrightarrow{com(m')} F_1(m') \circ N(o_2)$$

and another 2-morphism

$$N(o_1) \circ F_2(m) \xrightarrow{com(m)} F_1(m) \circ N(o_2) \xrightarrow{F_1(c) \circ_h id} F_1(m') \circ N(o_2),$$

and these 2-morphisms are required to coincide.

Definition 2.5. Let F_1, F_2 be a pair of functors between relative bicategories $\mathcal{C}, \mathcal{C}'$. A natural transformation first assigns a natural transformation between functors of associated categories of boundary transformations. Then it assigns a 1-morphism $N(o) : F_1(o) \rightarrow F_2(o)$ for every $o \in \mathcal{C}_0$, and a 2-morphism $com(m)$ for every $m : o_1 \rightarrow o_2$ which is an isomorphism

$$(2.2) \quad com(m) : N(o_1) \circ_1 F_2(m) \rightarrow F_1(m) \circ_1 N(o_2).$$

Naturality of $\text{com}(m)$ in the relative case is expressed as follows. There is an assigned a unital functor

$$TR : D \rightarrow \mathcal{C}'_1$$

s.t. $TR(o) = N(o)$ and $TR(d) : N(o) \rightarrow N(o')$ has L/R boundary transformations $F_1(d), F_2(d)$. Here D is the category of boundary transformations in \mathcal{C} and \mathcal{C}'_1 as usual denotes the category of 1-morphisms in \mathcal{C}' . TR is required to satisfy the following property. Given $m : o_1 \rightarrow o_2$, $m' : o'_1 \rightarrow o'_2$ and $c : m \rightarrow m'$ in \mathcal{C}_2 , we have a 2-morphism (with \circ_h denoting horizontal composition of 2-morphisms)

$$N(o_1) \circ F_2(m) \xrightarrow{TR(d_1) \circ_h F_2(c)} N(o'_1) \circ F_2(m') \xrightarrow{\text{com}(m')} F_1(m') \circ N(o'_2)$$

and another 2-morphism

$$N(o_1) \circ F_2(m) \xrightarrow{\text{com}(m)} F_1(m) \circ N(o_2) \xrightarrow{F_1(c) \circ_h TR(d_2)} F_1(m') \circ N(o'_2),$$

and these 2-morphisms are required to coincide.

Clearly the definition in the relative case reduces to the usual one if we think of a bicategory as a relative bicategory in the trivial sense: the category D has only identity morphisms.

3. RELATIVE BICATEGORIES OF OPEN CLOSED STRINGS IN X , AND DG-CATEGORIES

3.1. The relative bicategory \mathcal{V} of small differential graded categories. A (unital) differential graded category A over a field k is category enriched over the monoidal category Ch of chain complexes of k -vector spaces. This means that morphisms sets $A(a, b)$ are chain complexes over k , and composition is described by a map of chain complexes

$$A(a, b) \otimes A(b, c) \rightarrow A(a, c),$$

for the standard chain complex structure on the tensor product on the left. For example consider the differential graded category of chain complexes Ch_{dg} , with degree n morphisms $Ch_{dg, n}(C_1, C_2)$ being graded vector space maps $C_1 \rightarrow C_2[n]$, not necessarily preserving the differential. The differential on $Ch_{dg, n}(C_1, C_2)$ is

$$Df = f \circ d_{C_1} + (-1)^{n+1} d_{C_2} \circ f,$$

with d_{C_i} denoting the differential on C_i . Assume from now on $k = \mathbb{C}$.

An $A - B$ bimodule over a pair of dg-categories A, B is a Ch -enriched functor

$$(3.1) \quad V : A^{op} \otimes B \rightarrow Ch_{dg},$$

i.e. it is a functor preserving all structure, where A^{op} denotes the opposite category. In practice this means that we have a chain complex $V(a, b)$ for $a \in A$, $b \in B$ and for a pair of morphisms $c \rightarrow a$, $b \rightarrow d$ in degree n respectively m a degree $n+m$ map $V(a, b) \rightarrow V(c, d)$, which is functorial in all variables. What we call bimodule is also sometimes called a profunctor or distributor. Our particular choice of name is meant to further emphasize the formal connection to Von-Neuman algebra bimodules in [7] and Connes fusion.

Notation 3.1. We will write $\text{Hom}(A)$ for the $A - A$ bimodule

$$(a, b) \mapsto \text{Hom}_A(a, b).$$

We then have the following relative bicategory \mathcal{V} . The objects of \mathcal{V} are dg-categories. The 1-morphisms from A to B are $A - B$ bimodules, boundary transformations $D(A, B)$ are just dg-functors f from A to B . Given $V_1 \in \mathcal{V}_1(A, B)$, $V_2 \in \mathcal{V}_1(C, D)$ a 2-morphism of degree n from V_1 to V_2 is a pair of boundary transformations $f_L : A \rightarrow C$, $f_R : B \rightarrow D$ and a degree n natural transformation N from $V_1 : A^{op} \times B \rightarrow Ch_{dg}$ to $(f_L^{op} \otimes f_R) \circ V_2 : C \otimes D \rightarrow Ch_{dg}$, (we remind the reader that we are using diagrammatic order for composition.) Where by degree n we mean that all the structure maps

$$N(a, b) : V_1(a, b) \rightarrow f_L^{op} \otimes f_R \circ V_2(a, b)$$

have degree n . From this point of view the space 2-morphisms has the structure of a chain complex, with the obvious differential induced by D .

We have to say now how to define the composition of 1-morphisms in \mathcal{V} . There is an established theory of categorical bimodules, and their composition or “tensor product” which is a unital and associative operation (up to natural transformation) defined via left Kan extension or coend, see for example [1]. This is our composition, however we are not going to describe this abstraction instead opting to give an explicit description of the composition. This description corresponds to the abstract definition provided we consider our functors (3.1) as enriched over the $(\infty, 1)$ -category of chain complexes, in particular the composition is associative and unital up to natural transformations. We will still just write Ch for this $(\infty, 1)$ -category. The background for this latter point of view is in Lurie’s books [4, Section 4.3], [3]. We note also that the explicit definition we give, which is a kind of bar construction, is very natural from the point of view of the Donaldson-Floer-Fukaya TCFT object, as previously alluded to in the introduction.

Given $V_1 \in \mathcal{V}_1(A, B)$, and $V_2 \in \mathcal{V}_1(B, C)$, $V_1 \circ V_2(a, c)$ is defined as the total complex of the bigraded chain complex, which in degree $(n, k - 1)$ is

$$\bigoplus_{\substack{k\text{-tuples} \\ b_k, \dots, b_1 \in B}} (V_1(a, b_k) \otimes B(b_k, b_{k-1}) \otimes \dots \otimes B(b_2, b_1) \otimes V_2(b_1, c))_n,$$

where the subscript n denotes the degree n component of the tensor product. The pair of commuting differentials are given by the natural differential on the tensor product of the chain complexes in the above expression, and

$$\begin{aligned} d_H &= \sum_{i=0}^k (-1)^i d_i, \\ d_0(v_1 \otimes m_{k-1} \dots \otimes m_1 \otimes v_2) &= (v_1 m_{k-1}) \otimes m_{k-2} \otimes \dots \otimes v_2, \\ d_i(v_1 \otimes m_{k-1} \dots \otimes m_1 \otimes v_2) &= v_1 \otimes m_{k-1} \otimes \dots \otimes m_i m_{i-1} \otimes \dots \otimes v_2, \\ d_k(v_1 \otimes m_{k-1} \otimes \dots \otimes m_1 \otimes v_2) &= v_1 \otimes m_{k-1} \dots \otimes m_1 v_2. \end{aligned}$$

for

$$v_1 \otimes m_{k-1} \dots \otimes m_1 \otimes v_2 \in V_1(a, b_k) \otimes B(b_k, b_{k-1}) \otimes \dots \otimes B(b_2, b_1) \otimes V_2(b_1, c),$$

where $v_1 m_{k-1}$ comes from the right action of B on $V_1(a, b_k)$, $m_1 v_2$ comes from left action of B on $V_2(b_1, c)$, and the other contractions are just compositions in B . It can be readily verified that $d_H \circ d_H = 0$.

3.1.1. *Monoidal structure on \mathcal{V} .* This is the structure given by tensor product on objects, and the exterior tensor product on 1-morphisms, with the later being defined as follows: for $V_1 \in \mathcal{V}_1(A, B)$, $V_2 \in \mathcal{V}_1(C, D)$,

$$\begin{aligned} V_1 \otimes V_2 &\in \mathcal{V}_1(A \otimes C, B \otimes D), \\ V_1 \otimes V_2(a \otimes b, c \otimes d) &= V_1(a, c) \otimes V_2(b, d). \end{aligned}$$

The unit is \mathbb{C} , the dg-category with one object and morphism space just being \mathbb{C} with its multiplication for composition, graded in degree 0.

Adjunctions. We have a natural functor

$$Adj : \mathcal{V}_1 \rightarrow \mathcal{V}_1$$

by interpreting an $A - B$ bimodule as a $C - A^{op} \otimes B$ bimodule, where as usual \mathcal{V}_1 denotes the associated category of 1-morphisms in the relative bicategory.

3.2. **Open-closed string bicategory $OC(X)$.** We first give a preliminary definition. Suppose for the moment $X = pt$, then this is the relative bicategory whose objects are 0-dimensional oriented manifolds o , boundary transformations and 1-morphisms from o_1 to o_2 are either oriented diffeomorphisms from o_1 to o_2 or oriented 1-manifolds, together with diffeomorphism of the boundary to $o_1^{op} \sqcup o_2$, with o_1^{op} denoting o_1 with the opposite orientation.

A 2-morphism is either a diffeomorphism between underlying 1-manifolds or an equivalence class of a conformal surface with corners, with parametrized boundary, with some oriented boundary components labeled as *left/right boundary transformations*, which we sometimes abbreviate by calling them L/R-components. The boundary transformations may contain diffeomorphisms, these just correspond to corners between components corresponding to incoming/outgoing 1-morphisms. (These are not labeled.)

Two such conformal surfaces are called *equivalent* if they are related by a conformal diffeomorphism, identifying the boundary parametrizations, as well as the left/right boundary transformation labels. If an equivalence class of Σ is in the morphism space $OC_2(m_1, m_2)$, $m_1 \in OC_1(a, b)$, $m_2 \in OC_1(c, d)$ relative to boundary transformations d_L, d_R then it is furnished with a diffeomorphism of the boundary of Σ to $d_L \sqcup d_R^{op} \sqcup m_1^{op} \sqcup m_2$, with m_1^{op} denoting m_1 with opposite orientation, and with L/R components going to d_L , respectively d_R . The structure functors $L : OC_1 \rightarrow D$, $R : OC_1 \rightarrow D$ are then obviously induced by the left/right boundary transformations. While the composition operations are the natural gluing operations of oriented 1-manifolds, and conformal surfaces with parametrized boundary. For a general CW-complex X we add continuous target maps into X for all our objects, morphisms, and boundary transformations, and this gives us a naturally topological relative bicategory. More explicitly: the objects, 1-morphisms and 2-morphisms of $OC(X)$ are pairs (o, f) , (m, g) , respectively equivalence classes of pairs (Σ, h) for o, m, Σ objects, 1-morphisms respectively representatives for 2-morphisms in $OC(pt)$ and f, g, h continuous maps of underlying manifolds into X . A pair of 2-morphisms (Σ_1, h_1) , (Σ_2, h_2) are said to be equivalent if Σ_1, Σ_2 represent the same morphism in $OC(pt)$ and the corresponding conformal diffeomorphism $g : \Sigma_1 \rightarrow \Sigma_2$ satisfies: $h_2 \circ g = h_1$. The pair (Σ, f) represents a morphism in $OC_2(X)((m_1, g_1), (m_2, g_2))$ if Σ represents a morphism in $OC_2(pt)(m_1, m_2)$ and restriction of f to boundary components m_1, m_2 (via identification) is g_1 , respectively g_2 .

The Mayer-Vietoris property will actually require a slight technical enlargement of the category $OC(X)$. Again for $X = pt$, this is done by keeping the same objects and boundary transformations and defining 1-morphisms from o to o' to be oriented smooth 1-manifolds m with marked corners, with some components between marked corners labeled as *free*, with diffeomorphism of the boundary of m to $o^{op} \sqcup o'$. For a given 1-morphism, the complement of the associated free components is called *principal*, and this is assumed to be a smooth 1-manifold.

We extend this as before for a general X . Then a pair of 1-morphisms

$$(m_1, f_1), (m_2, f_2) \in OC_1(X)$$

will be said to be equivalent if m_i are related by a diffeomorphism g preserving all labels, s.t. $f_2 \circ g = f_1$ on principal components. The composition of 1-morphisms is defined by smooth gluing along boundary, i.e. no new corners are created. This should be clarified by the diagram for composition in Figure 1. Intuitively speaking free components are just built in identifications of objects.

The 2-morphisms are much as before, but with extra structure of new labels on some of the smooth edges of the underlying conformal surface. For $\Sigma \in OC_2(m_1, m_2)$, $m_1 : o_1 \rightarrow o_2$, $m_2 : o_2 \rightarrow o_3$ relative to boundary transformations d_L, d_R , the boundary of Σ as before comes with a diffeomorphism to $d_R^{op} \sqcup d_L \sqcup m_1^{op} \sqcup m_2$ and in particular there are smooth components in the boundary labeled as *free* or just F .

Vertical composition of 2-morphisms is defined by gluing along principal parts of the 1-morphisms as defined above and likewise horizontal composition is defined by gluing along principal parts of L/R components. We illustrate an example in Figure 1.

Monoidal structure on $OC(X)$. This is given by disjoint union on the underlying geometric objects.

3.2.1. Adjunction and free transformations. Recall that $OC_1(X)$ denotes the associated category of 1-morphisms in the relative bicategory. We then have a natural functor

$$Adj : OC_1(X) \rightarrow OC_1(X),$$

by interpreting a 1-morphism from a to b as a morphism from \emptyset to $a^{op} \sqcup b$.

Aside from the adjunction, we also have certain transformations of 2-morphisms which we call *free transformation*. Given a connected component d_0 of $L(\Sigma)$, or of $R(\Sigma)$ for Σ a given 2-morphism from m_1 to m_2 , we remove d_0 , instead considering it as a free component of a new morphism $fr(m_2)$, obtained from m_2 by “adjoining” to it the new free component. In this way given $\Sigma \in OC_2(m_1, m_2)$ we get a new 2-morphisms $fr(\Sigma) \in OC_2((m_1, fr(m_2)))$. Exactly what “adjoining” means, is completely determined by the condition that $fr(\Sigma)$ is a 2-morphism relative to the new left/right boundary transformations. We could do a similar thing with m_1 , in practice though we only need to do this procedure with just the target morphism.

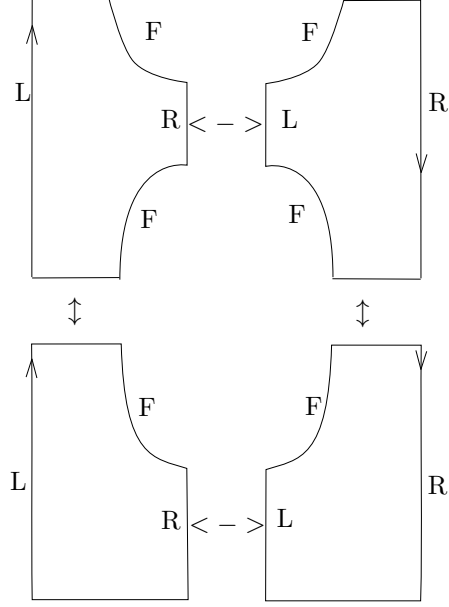


FIGURE 1. Diagram for vertical/horizontal composition. The components marked F are free. In this case there is only one order for composition due to the equivalence relation on 1-morphisms: we first compose top and bottom pieces horizontally and then compose vertically.

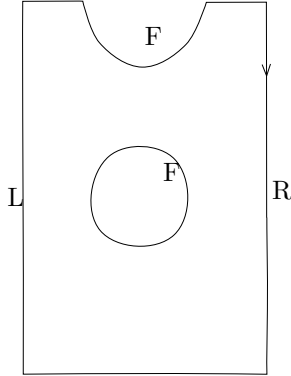


FIGURE 2. Diagram for the composed 2-morphism.

The differential graded bicategory $\mathcal{OC}(X)$.

Remark 3.2. *The most natural next step in our context seems to be to construct some kind of $(\infty, 2)$ -category from the topological relative bicategory $\mathcal{OC}(X)$. A related possibility is to apply the singular chains functor to the topological category $\mathcal{OC}(X)$, this however does not give a differential graded relative bicategory in the*

expected sense as the space of objects itself becomes a chain complex. Both of these possibilities, in the context of TCFD objects (particularly in the context of Mayer-Vietoris property) present a number of technical and conceptual challenges, and we choose to postpone this development for the future. Instead we will work with a simplest possible approximation of the second possibility, which however already has a very rich structure.

We associate to $OC(X)$ a differential graded relative bicategory $\mathcal{OC}(X)$, with the same set of objects, 1-morphisms, same collection of boundary transformations D , but defining 2-morphisms in $\mathcal{OC}_2(X)(m_1, m_2)$ to be singular chains in $OC_2(X)(m_1, m_2)$, which are constant as chains in the mapping space of the topological surface Σ . More precisely, denote by $OC_2^{top}(X)$, the quotient space of $OC_2(X)$ obtained by forgetting the conformal structure in the equivalence relation, and $fog : OC_2(X) \rightarrow OC_2^{top}(X)$ the quotient map. Then morphisms in $\mathcal{OC}_2(X)(m_1, m_2)$ are singular chains in $OC_2(X)(m_1, m_2)$ with image in a fiber of fog , and consequently $\mathcal{OC}_2(X)(m_1, m_2)$ has a grading and a differential D_{ch} . Although the way we define it, it only has a partial additive structure. The grading will be denoted by an extra subscript: $\mathcal{OC}_{2,k}(m_1, m_2)$. Note that $\mathcal{OC}(X)$ also inherits the monoidal structure, the adjunction transformation from $OC(X)$, and the free transformations.

The category $\mathcal{OC}(X)$ has a subcategory $\mathcal{C}(X)$ whose objects are the null objects of $\mathcal{OC}(X)$ (empty 0-manifolds), 1-morphisms are closed oriented 1-manifolds and associated 2-morphisms restricted to have no free components.

Notation 3.3. *The 1-morphisms of \mathcal{C} will be denoted by their number of components n , whenever orientation of components is not explicitly relevant. A 2-morphism in $\mathcal{C}_{2,k}(m, n)$, corresponding to a chain in the component of $\mathcal{OC}_{2,k}$, where the underlying surface has euler characteristic χ , will be denoted by $c_{k,\chi}$.*

We say that two morphisms in $\mathcal{OC}_{2,k}(X)(m_1, m_2)$ are *homotopic* if the underlying chains are homotopic in $OC_2(m_1, m_2)$. (We do mean homotopic and not homologous.) Define $ho\mathcal{OC}(X)$ to be the category with same objects, and 1-morphisms, with 2-morphisms: homotopy classes of closed chains underlying morphisms in $\mathcal{OC}_2(m_1, m_2)$ and identifying closed chains homologous in the fiber of $fog : OC_2(X) \rightarrow OC_2^{top}(X)$. And likewise boundary transformations $D(o_1, o_2)$ are replaced by homotopy classes (relative o_1, o_2) of the underlying maps.

Define also $ho\mathcal{V}$ as the category with same objects, and 1-morphisms and taking the 2-morphism space $ho\mathcal{V}_2(m_1, m_2)$ to be the homology groups of $\mathcal{V}_2(m_1, m_2)$ with respect to the differential D , cf. (3.1). The boundary transformations in $ho\mathcal{V}$ are functors of dg -categories enriched over the homotopy category of Ch , again considered as an $(\infty, 1)$ -category. The latter is just the derived category $D(Vect)$.

Invariance under free transformations. Given $c \in \mathcal{OC}_2(m_1, m_2)$ and $fr(c)$ the 2-morphism given by adjoining some collections l, r of left/right boundary components as free components to $m_2 : o_{2,L} \rightarrow o_{2,R}$, we have induced bordisms, which we think of boundary transformations

$$d_l : o_{2,l} \rightarrow o_{2,L},$$

$$d_r : o_{2,r} \rightarrow o_{2,R}$$

for some $o_{2,l}, o_{2,r}$ determined by l, r .

Definition 3.4. For an $F : \mathcal{OC}(X) \rightarrow \mathcal{V}$, we say that it is invariant under free transformations if there is a 2-morphism

$$T : F(fr(m_2)) \rightarrow F(m_2),$$

relative to $F(d_l)$, $F(d_r)$, such that:

$F(c) : F(m_1) \rightarrow F(m_2)$ is the composition

$$F(m_1) \xrightarrow{fr(c)} F(fr(m_2)) \xrightarrow{T} F(m_2).$$

Loosely speaking all this means is that our technical enlargement of 1-morphism and 2-morphism space is algebraically only formal.

The name Calabi-Yau below is motivated by Costello's definition of Calabi-Yau categories in [2]. Since for $X = pt$, the data of a Calabi-Yau functor has something to do with a dg-category intertwined with an open-closed TCFT. Note however that we do not explicitly require Calabi-Yau condition on our dg-categories. Still it seems very likely that for $X = pt$ there is a connection between the notions.

We note also that all of the following formal properties except the grading condition are actually used for the proof of Mayer-Vietoris property later on.

Definition 3.5. A functor of bicategories $F : \text{ho } \mathcal{OC}(X) \rightarrow \text{ho } \mathcal{V}$ is said to be Calabi-Yau of degree d , if:

- It respects the involution op_0 , taking the adjunction to the adjunction, and is invariant under free transformations.
- For $0 \neq c \in \mathcal{OC}_{2,0}((m, f), (m, f))$ a 2-morphism whose underlying surface is diffeomorphic to $m \times [0, 1]$ for m a 1-manifold with boundary, the associated map $\text{ho } F(c)$, is an isomorphism.
- F is h -monoidal: there are isomorphisms in $\text{ho } \mathcal{V}$.

$$\text{ho } F(A \otimes B) \rightarrow \text{ho } F(A) \otimes F(B),$$

$$\text{ho } F(m_1 \otimes m_2) \rightarrow \text{ho } F(m_1) \otimes \text{ho } F(m_2), \text{ for } m_1, m_2 \in \mathcal{OC}_1(X)$$

- This merely fixes grading: for

$$c_{k,\chi} \in \mathcal{C}_{2,k}(n, m),$$

$$F(c_{k,\chi}) \in \mathcal{V}_{2,k-d(\chi+m-n)}.$$

Whitney sum on functors $\mathcal{OC}(X) \rightarrow \mathcal{V}$. For a pair of degree d TCFT objects F_1, F_2 , there is an object $F_1 \oplus F_2$, defined on objects by

$$F_1 \oplus F_2(z) = F_1(z) \oplus F_2(z),$$

where $F_1(z) \oplus F_2(z)$ is a dg-category with objects (a, b) for $a \in F_1(z)$, $b \in F_2(z)$ and with morphism space $F_1(z) \oplus F_2(z)((a, b), (c, d)) = F_1(a, c) \oplus F_2(b, d)$, for the usual sum operation on chain complexes.

Similarly if $m \in \mathcal{OC}_1(z_1, z_2)$ $F_1 \oplus F_2(m)$ is the $F_1 \oplus F_2(z_1) - F_1 \oplus F_2(z_2)$ bimodule defined by

$$F_1 \oplus F_2(m)((a, b), (c, d)) = F_1(m)(a, c) \oplus F_2(m)(b, d).$$

And this obviously extends to 2-morphisms.

Lemma 3.6. Given degree d Calabi-Yau functors F_1, F_2 their sum $F_1 \oplus F_2$ is also Calabi-Yau of degree d .

The proof is immediate from definitions. Note that from this point of view degree does not behave like rank of vector bundles, which of course is what we want if degree is to reflect cohomological degree in the following section.

4. TCFT OBJECTS

We first describe certain normalization conditions which may be omitted on a first reading.

Definition 4.1. We say that $V \in \mathcal{V}_2(A, B)$ is **induced** by $F : A \rightarrow B$, if $V(a, b) = \text{Hom}_B(F(a), b)$.

Definition 4.2. A Calabi-Yau functor $F : \text{ho } \mathcal{OC}(X) \rightarrow \text{ho } \mathcal{V}$ is said to be **normalized** if the following holds:

- For an invertible 1-morphism $m : o_L \rightarrow o_R$, the $F(o_L) - F(o_R)$ bimodule $F(m)$ is induced by $F_{bd}(m)$, considering m as a boundary transformation. In particular we have a tautological vertically invertible 2-morphism

$$c_m : F(m) \rightarrow \text{Hom}(F(b)).$$

- Given $c \in \mathcal{OC}_{2,0}(m_1, m_2)$, relative to boundary transformations $d_L : o_L \rightarrow o'_L$, $d_R : o_R \rightarrow o'_R$, with m_i invertible and c horizontally invertible:

$$(4.1) \quad F(c) = c_{m_1} \circ \text{id}(F_{bd}(d_R)) \circ c_{m_2}^{-1} \in \text{ho } \mathcal{V}_2(F(m_1), F(m_2)),$$

with \circ denoting vertical composition, and $\text{id}(F_{bd}(d))$ denoting horizontal identity 2-morphism relative to $F_{bd}(d_R)$, which is just the 2-morphism

$$\text{id}_{d_R} \in \mathcal{V}_2(\text{Hom}(F(o_R), \text{Hom}(F(o'_R)))),$$

naturally induced by the functor $F_{bd}(d_R) : F(o_R) \rightarrow F(o'_R)$.

Definition 4.3. We shall say that a natural isomorphism N between functors $F_1, F_2 : \text{ho } \mathcal{OC}(X) \rightarrow \text{ho } \mathcal{V}$ is **normalized** if:

- The associated functor $TR : D \rightarrow \text{ho } \mathcal{V}_1$ is a monoidal functor, where D as usual denotes the category of boundary transformations in the source category. (These are monoidal categories with respect to disjoint union respectively exterior tensor product.)
- For every object a , the bimodule $N(a)$ is induced by some isomorphism $I_N(a) : F_1(a) \rightarrow F_2(a)$ with $I_{N^{-1}}(a) = I_N^{-1}(a)$. In particular we have a tautological vertically invertible 2-morphism $c_{N,a} : N(a) \rightarrow \text{Hom}(F_2(a))$. Then given a boundary transformation $d : a \rightarrow b$ we have:

$$(4.2) \quad TR(d) = c_{N,a} \circ \text{id}(F_2(d)) \circ c_{N,b}^{-1} \in \text{ho } \mathcal{V}_2(N(a), N(b)),$$

with \circ denoting vertical composition.

Definition 4.4. A TCFT object of degree d on X is defined as a normalized degree d Calabi-Yau 2-functor $\text{ho } \mathcal{OC}(X) \rightarrow \text{ho } \mathcal{V}$.

We define $\mathcal{F}_d(X)$, to be the Grothendieck group completion of the Abelian monoid of equivalence classes of degree d TCFT objects on X , under normalized natural isomorphism. And we define reduced groups $\tilde{\mathcal{F}}_d(X) = \mathcal{F}_d(X) / \text{const}^* \mathcal{F}_d(pt)$, for $\text{const} : X \rightarrow pt$.

In addition to Whitney sum, we may completely analogously also take tensor products of Calabi-Yau functors and this operation is additive on degree. Consequently we get a certain graded ring $\mathcal{F}_*(X)$, with graded summands $\mathcal{F}_d(X)$. It

would be very interesting to identify $\mathcal{F}_*(pt)$, and as we already mentioned Costello's [2] may have relevance to this.

Theorem 4.5.

$$\begin{aligned}\tilde{\mathcal{F}}_d : Top &\rightarrow Ab, \\ X &\mapsto \tilde{\mathcal{F}}_d(X)\end{aligned}$$

is a representable cofunctor, where Top denotes the homotopy category of based topological spaces and Ab denotes the category of Abelian groups, in other words we have that

$$\tilde{\mathcal{F}}_d(X) = [X, \mathcal{R}_d],$$

for some $\mathcal{R}_d \in Top$ an H -space.

Proof. By the celebrated Brown representability theorem this will follow once we show that:

- The pullback maps $f^* : \tilde{\mathcal{F}}_d(Y) \rightarrow \tilde{\mathcal{F}}_d(X)$ depend only on the homotopy class of $f : X \rightarrow Y$, i.e. $\tilde{\mathcal{F}}_d$ is a homotopy functor.
- The Mayer-Vietoris property is satisfied: for $X = U \cup V$ the sequence

$$\tilde{\mathcal{F}}_d(X) \rightarrow \tilde{\mathcal{F}}_d(U) \oplus \tilde{\mathcal{F}}_d(V) \rightarrow \tilde{\mathcal{F}}_d(U \cap V),$$

is exact.

- $\tilde{\mathcal{F}}_d$ takes coproducts to products, i.e.

$$\tilde{\mathcal{F}}_d\left(\bigvee_{\alpha} X_{\alpha}\right) = \prod_{\alpha} \tilde{\mathcal{F}}_d X_{\alpha}.$$

We need to show that if $f, g : X \rightarrow Y$, are homotopy equivalent then $F_f = f^*F$ is equivalent to $F_g = g^*F$, for F a degree d TCFT object on Y . Let $o_1, o_2 \in \mathcal{OC}_0(X)$, then a homotopy $H : X \times [0, 1] \rightarrow Y$ between f, g gives us morphisms $n^t(o_1), n^t(o_2)$ in $\mathcal{OC}_1(f_*o_1, H_*^t o_1)$, respectively $\mathcal{OC}_1(g_*o_2, H_*^t o_2)$. And for $m \in \mathcal{OC}_1(o_1, o_2)$, the homotopy H induces a 2-morphism in $\mathcal{OC}_2(X)$,

$$(4.3) \quad com_H(m) : n(o_1) \circ g_*(m) \rightarrow f_*m \circ n(o_2),$$

where $n(o_1) = n^1(o_1)$ and $n(o_2) = n^1(o_2)$. For $d : o \rightarrow o'$ a boundary transformation, we have a 2-morphism $tr(d) : n(o) \rightarrow n(o')$, with left/right boundary transformations f_*d, g_*d : it is simply the track by the homotopy H of d . Now apply F to (4.3) and set $N(o) = F(n(o))$, then we get a 2-morphism

$$com(m) : N(o_1) \circ F_2(m) \rightarrow F_1(m) \circ N(o_2).$$

We define $TR(d)$ of Definition 2.5 as $F(tr(d))$. We need to now explain naturality of 2-morphisms $com(m)$.

Given $c : m \rightarrow m'$, for $m : o_1 \rightarrow o_2$, $m' : o'_1 \rightarrow o'_2$, with left right boundary transformations $d_1 : o_1 \rightarrow o'_1$, $d_2 : o_2 \rightarrow o'_2$, the chains corresponding to the pair of morphisms in $\mathcal{OC}_2(X)$:

$$(4.4) \quad n(o_1) \circ g_*m \xrightarrow{tr(d_1) \circ_h g_*c} n(o'_1) \circ g_*(m') \xrightarrow{com_H(m')} f_*m' \circ n(o'_2)$$

$$(4.5) \quad n_1(o_1) \circ g_*m \xrightarrow{com_H(m)} f_*m \circ n(o_2) \xrightarrow{(f_*c) \circ_h tr(d_2)} f_*m' \circ n(o'_2),$$

are homotopic in $\mathcal{OC}_2(X)$. We now explain this. We have a 2-morphism

$$com_t : n(o_1) \circ g_*(m) \rightarrow n^t(o_1) \circ H_*^t m \circ n^t(H_*^t o_2),$$

induced by com_H , where $n^t(H_*^t o_2) : H_*^t o_2 \rightarrow g_* o_2$, is induced by H . Then we have a 2-morphism

$$tr_t : n^t(o_1) \circ H_*^t m \circ n^t(H_*^t o_2) \rightarrow n^t(o'_1) \circ H_*^t m' \circ n^t(H_*^t o'_2),$$

which is just the horizontal composition of 2-morphisms $tr : n^t(o_1) \rightarrow n^t(o'_1)$, $H_*^t c : H_*^t m \rightarrow H_*^t m'$, $tr : n^t(H_*^t o_2) \rightarrow n^t(H_*^t o'_2)$. Finally, we have a 2-morphism induced by com_H

$$com'_t : n^t(o'_1) \circ H_*^t m' \circ n^t(H_*^t o'_2) \rightarrow f_* m' \circ n(o'_2).$$

Then the family of 2-morphisms $com_t \circ tr_t \circ com'_t$ gives a homotopy between (4.4), (4.5), and applying the functor $ho F$ to these equations we get the desired naturality for com . The corresponding natural transformation is normalized, which is immediate from construction and from the assumption that F is normalized, consequently F_f is equivalent to F_g .

We now verify Mayer-Vietoris property. Denote an equivalence class of a TCFT object F by $|F|$. For a pair of degree d objects F_U, F_V on U , respectively V , with $|F_U| - |F_V| = 0 \in \tilde{\mathcal{F}}(X)$, we need to construct an object F on X , that restricts to objects equivalent to F_U, F_V on U, V . By assumption $|F_U| - |F_V| = |T|$, for $|T| \in const^* \mathcal{FT}_d(pt) \subset \mathcal{FT}_d(U \cap V)$. However, we may just as well suppose that $|T| = 0$, as we can set $F'_V = F_V + T$, where T is the obvious extension of T on $U \cap V$ to V . Then $|F_U| - |F'_V| = 0$, and $|F'_V| = |F_V|$.

Lemma 4.6. *There is a TCFT object \tilde{F}_V on V , equivalent to F_V and coinciding with F_U on $U \cap V$.*

Proof. Let N denote the natural isomorphism from F_V to F_U on $\mathcal{OC}(U \cap V)$. Define \tilde{F}_V on objects by $\tilde{F}_V(o) = F_V(o)$ for $o \in V - U$, and $\tilde{F}_V(o) = F_U$ for $o \in V \cap U$. For a 1-morphism $m_1 : o_L \rightarrow o_R$, $o_L \in V - U$, $o_R \in V \cap U$, $\tilde{F}_V(m_1) = F_V \circ N(o_R)$. If on the other hand for m_2 , $o_L \in V \cap U$, $o_R \in V - U$ define $\tilde{F}_V(m_1) = N_{o_L}^{-1} \circ F_V(m_2)$.

For $m_3 \in \mathcal{OC}_1(V \cap U)$, $\tilde{F}_V(m_3) = F_U(m_3)$. Finally, for $m_4 \in \mathcal{OC}_1(V - U)$ define $\tilde{F}_V(m_3) = F_V(m_3)$. For a general m first decompose it as composition of types above, and define $\tilde{F}_V(m)$ by functoriality.

Define \tilde{F}_V similarly on boundary transformations. For a composable pair m_1, m_2 with these 1-morphisms as above, define the naturality 2-morphism (see (2.1))

$$(4.6) \quad Nat : \tilde{F}_V(m_1 \circ m_2) \rightarrow \tilde{F}_V(m_1) \circ \tilde{F}_V(m_2),$$

by $Nat = id \circ_h com_N(m_2)$ where id is the identity $m_1 \rightarrow m_1$ and $com_N(m_2)$ is the commutation 2-morphism induced by the natural transformation N , see (2.2). Similarly for a composable pair m_2, m_3 again with these 1-morphisms of type above, $Nat = com_{N^{-1}}(m_2) \circ_h id$. In other cases just define Nat to be identity.

We need to define $\tilde{F}_V(c)$ separately for c with various types of boundary data. We won't write out every possibility, as they are just doing the only thing which can be done, and so are somewhat self explanatory, but here are a few. Suppose we are given a 2-morphism $c \in \mathcal{OC}_2(V)(m_1, m_2)$, $m_1 : o_{1,L} \rightarrow o_{1,R}$, $m_2 : o_{2,L} \rightarrow o_{2,R}$, with relative boundary transformations $d_l : o_{1,L} \rightarrow o_{2,L}$, $d_r : o_{1,R} \rightarrow o_{2,R}$.

Case 1: suppose that $o_{1,L}, o_{2,L} \in V - U$ and $o_{1,R}, o_{2,R} \in U \cap V$, then define $\tilde{F}_V(c_1)$ as the horizontal composition $F(c_1) \circ_h TR_N(d_r)$, note that the expression $TR_N(d_r)$ makes sense even if d_r does not lie in $U \cap V$, if we define it via (4.2). (In our case $F_1 = F_V$, $F_2 = \tilde{F}_V$, $a = o_{1,R}$, $b = o_{2,R}$.)

Case 2: $o_{2,L}, o_{2,R}, o_{1,L}$ are as before but $o_{1,R}$ lies in $V - U$ then $\tilde{F}_V(c)$ is defined as

$$\tilde{F}_V(c) = F_V(c) \circ_h (id(d_R) \circ c_{N^{-1}, o_{2,R}}),$$

in the notation of Definition 4.3.

Case 3: if $o_{i,L}, o_{i,R}$ lie in $U \cap V$, then define

$$\tilde{F}_V(c) = F_U(c).$$

Case 4: $o_{2,L}, o_{2,R}$ lie in $U \cap V$ and $o_{1,L}, o_{1,R}$ lie in $V - U$. Then define

$$\tilde{F}_V(c) = \left((id(d_L) \circ c_{N^{-1}, o_{2,L}}^{-1}) \circ_h F_V(c) \circ_h (id(d_R) \circ c_{N, o_{2,R}}^{-1}) \right) \circ com'_N(m_2),$$

where $com'_N(m_2) : N^{-1}(o_{2,L}) \circ F_V(m_2) \circ N(o_{2,R}) \rightarrow F_U(m_2)$, is the composition of $id \circ_h com_N(m_2) : N^{-1}(o_{2,L}) \circ F_V(m_2) \circ N(o_{2,R}) \rightarrow N^{-1}(o_{2,L}) \circ N(o_{2,L}) \circ F_U(m_2)$, where $com_N(m_2)$ given by the natural transformation N , with the 2-morphism

$$N^{-1}(o_{2,L}) \circ N(o_{2,L}) \circ F_U(m_2) \rightarrow F_U(m_2).$$

The point of course is that \tilde{F}_V is in fact a functor, which amounts to first checking that \tilde{F}_V is a functor $\mathcal{OC}_1(V) \rightarrow \mathcal{V}_1$, which is essentially immediate from definitions and the defining property of the functor TR_N . And then checking that the above defined Nat gives a natural transformation commutator for

$$(4.7) \quad \begin{array}{ccc} \mathcal{OC}_1(V) \times_{L,R} \mathcal{OC}_1(V) & \longrightarrow & \mathcal{OC}_1 \\ \downarrow & & \downarrow \tilde{F}_V \\ \mathcal{V}_1 \times_{L_V, R_V} \mathcal{V}_1 & \longrightarrow & \mathcal{V}_1 \end{array}$$

see (2.1).

This is straightforward to check although again we won't right out every case. For example suppose we have a pair of horizontally composable 2-morphisms c_1, c_2 with c_1 of type 1 and c_2 of type 3, along $d : o \rightarrow o'$, (i.e. d is the right respectively left boundary transformation of c_1 respectively c_2). Then

$$(4.8) \quad \left(\tilde{F}_V(c_1) \circ_h \tilde{F}_V(c_2) \right) \circ Nat = F_V(c_1) \circ_h TR_N(d) \circ_h F_U(c_2) \circ Nat,$$

by the defining property of TR_N this is:

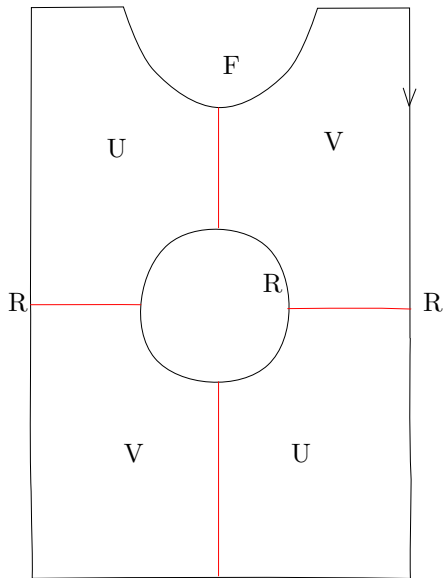
$$(4.9) \quad Nat \circ (F_V(c_1) \circ_h F_V(c_2) \circ_h TR_N(d_R)) = Nat \circ \left(\tilde{F}_V(c_1 \circ c_2) \right),$$

for d_R the right boundary transformation of c_2 .

Finally, the fact that \tilde{F}_V is equivalent to F_V is built into its definition. \square

Lemma 4.7. *Given \tilde{F}_V as in the above lemma, there is an induced object F on $U \cup V$, restricting to \tilde{F}_V, F_U over V , respectively U .*

Proof. On the level of objects and 1-morphisms it is clear how to proceed as this is highly analogous to K -theory (if we think of it in terms of equivalence classes of functors on the path category): decompose a 1-morphism in $\mathcal{OC}_1(X)$ as composition and or disjoint union of 1-morphisms in $\mathcal{OC}_1(U), \mathcal{OC}_1(V)$ and define F by functoriality and or h-monoidal property. For example suppose $m = m_U \circ m_V$ for $m_U : o_1 \rightarrow o_2, m_V : o_2 \rightarrow o_3$ 1-morphisms in $\mathcal{OC}_1(U), \mathcal{OC}_1(V)$. Define $F(m)$ by $F(m) = F_U(m_U) \circ \tilde{F}_V(m_V)$. The case of a general decomposition of a 1-morphism is similar.



Also for 2-morphisms coming from diffeomorphisms there is nothing to do. Given a general 2-morphism in $c \in \mathcal{OC}_2(X)$, we may also decompose it as horizontal and vertical compositions of morphisms in $\mathcal{OC}_2(U), \mathcal{OC}_2(V)$. The only tricky part of this is that we may need to use the adjunctions, and free transformations. In Figure 3 is illustrated a prototypical example. To assign value of F on this 2-morphism c , first use that F would be invariant under free transformations, and rename the circular R component as a free component. We may then use adjunction to decompose it in a way analogous to example in Figure 1 and proceed to define F by functoriality, as before.

$$\mathcal{U} : \tilde{\mathcal{F}}_d(X) \rightarrow \prod_{\alpha} \tilde{\mathcal{F}}_d(X_{\alpha}),$$
$$|i_\alpha^* F| = |\text{const}^* F_0| \in \mathcal{F}_d(X_\alpha),$$

for $i_\alpha : X_\alpha \rightarrow X$ and $const : X_\alpha \rightarrow pt$ based maps for some F_0 . The object F_0 is independent of α , since the based maps $pt \rightarrow X_\alpha \rightarrow X$ coincide, and $pt \rightarrow X_\alpha \rightarrow pt$ is identity so we must have $|F_0| = |i^*F| \in \mathcal{F}_d(pt)$ for $i : pt \rightarrow X$ the based map.

So we just need to show that it follows that $|F| = |G| \in \mathcal{F}_d(X)$, for $G = \text{const}^* F$, where $\text{const} : X \rightarrow X$ is the constant map to the based point x_0 .

Given $m \in \mathcal{OC}_1(X)(o_L, o_R)$ decompose it as a composition of 1-morphisms $m_i \in \mathcal{OC}_1(X)(o_{i,L}, o_{i,R})$, $1 \leq i \leq n$ with each m_i mapping to $X(\alpha_i)$, and $o_{i,L}, o_{i,R} \in \mathcal{OC}_0(x_0)$ except $o_L = o_{1,L}$ and $o_{n,R} = o_R$. Natural transformations N_i between F and G on X_{α_i} give 1-morphisms:

$$\begin{aligned} N_i(o_{i,L}) : F(o_{i,L}) &\rightarrow G(o_{i,L}), \\ N_i(o_{i,R}) : F(o_{i,R}) &\rightarrow G(o_{i,R}), \end{aligned}$$

and $F(o_{i,L}) = G(o_{i,L})$, $F(o_{i,R}) = G(o_{i,R})$, except at $o_{1,L}$ and $o_{n,R}$. Consequently, we may suppose that $N_i(o_{i,L}), N_{i-1}(o_{i-1,R})$ coincide except for $o_{1,L}, o_{n,R}$, whenever this makes sense in terms of indexes. As otherwise define $\tilde{N}_i = N_i \circ N_0$ for N_0 the natural transformation $G \rightarrow F$, $N_0(o) = N_i(o_{x_0})^{-1} \circ N_1(o_{x_0})$, where o_{x_0} is the object $pt \mapsto x_0 \in X$, and o some object $pt \rightarrow X_{\alpha_i}$. This is trivially a natural transformation by defining TR_{N_0} via (4.2). Then \tilde{N}_i have the above property.

But then the commutation 2-morphisms

$$\text{com}_i : N_i(o_i) \circ G(m_i) \rightarrow F(m_i) \circ N(o'_i)$$

horizontally compose to a commutation 2-morphism

$$\text{com} : N_1(o_1) \circ G(m) \rightarrow F(m) \circ N_n(o'_n).$$

We must show that it is natural. Given $c : m \rightarrow m'$ with m, m' decomposed as above, it is homotopic (after decomposing into finer pieces if necessary) to a horizontal composition of morphisms $c_i : m_i \rightarrow m'_i$, for some $m_i : o_{i,L} \rightarrow o_{i,R}$, $m'_i : z_{i,L} \rightarrow z_{i,R}$, in $\mathcal{OC}_1(X_{\alpha_i})$.

We first show that except for the extremal end points the following 2-morphisms coincide:

$$\begin{aligned} TR_i(d_{i,L}) : N_i(o_{i,L}) &\rightarrow N_i(z_{i,L}) \\ TR_{i-1}(d_{i-1,R}) : N_{i-1}(o_{i-1,R}) &\rightarrow N_{i-1}(z_{i-1,R}). \end{aligned}$$

Since our natural transformations are normalized,

$$\begin{aligned} TR_i(d_{i,L}) &= c_{N_i}(o_{i,L}) \circ \text{id}(d_{i,L}) \circ c_{N_i^{-1}}(z_{i,L}), \\ TR_{i-1}(d_{i-1,R}) &= c_{N_{i-1}}(o_{i-1,R}) \circ \text{id}(d_{i-1,R}) \circ c_{N_{i-1}^{-1}}(z_{i-1,R}). \end{aligned}$$

By assumptions $d_{i-1,R} = d_{i,L}$. On the other hand $c_{N_i}(o_{i,L}) = c_{N_{i-1}}(o_{i-1,R})$, since $N_i(o_{i,L}) = N_{i-1}(o_{i-1,R})$ and by the well known fact that if a Ch enriched functor $C_1^{op} \otimes C_2 \rightarrow Ch_{dg}$ is induced by a Ch enriched functor $F : C_1 \rightarrow C_2$ (as in definition 4.3), then F is uniquely determined.

Consequently, naturality of com follows from naturality of com_i provided each c_i lies in $\mathcal{OC}_2(X(\alpha_i))$. But if this is not satisfied we may decompose c_i as in the proof of Mayer-Vietoris and use functoriality to reduce to the previous case. \square

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